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89-FM-10

JSC-23762

A Fast Recursive Singularity Free Algorithm for Calculating the First and Second Derivatives of the Geopotential

Engineering Directorate
Navigation, Control, and Aeronautics Division

July 6, 1990

NASA
National Aeronautics and
Space Administration

Lyndon B. Johnson Space Center
Houston, Texas

89-FM-10 JSC-23762

SHUTTLE PROGRAM

A Fast Recursive Singularity Free Algorithm for Calculating the First and Second Derivatives of the Geopotential

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SECTION 1
SUMMARY

This report presents the development of a recursive, non-singular method for computing the first and second partials of the geopotential with respect to the position vector, $\underline{X}^T = (x_1 \; x_2 \; x_3)$. The magnitude of $\underline{X}$ is the radius, r.

The second partial, which is the partial derivative of the gravitational acceleration with respect to the position, is required in a number of disciplines; e.g., estimation, optimization, gravity gradient, etc.

A non-singular, recursive algorithm for computing the first and a symmetric, non-singular, recursive algorithm for computing the second derivative of the geopotential are derived, programmed, and verified.

SECTION 2
INTRODUCTION

This report presents the development of a recursive, non-singular method for computing the first and second partials of the geopotential with respect to the position vector, $\underline{X}^T = (x_1 \ \ x_2 \ \ x_3)$. The magnitude of $\underline{X}$ is the radius, r.

The second partial, which is the partial derivative of the gravitational acceleration with respect to the position, is required by a number of disciplines; e.g., estimation, optimization, gravity gradient, etc.

The gravitational potential function for the Earth is normally written

$$V = -\frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\mu}{r}\left(\frac{a_e}{r}\right)^n P_{n,m}(\varepsilon)\,(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda) \tag{1}$$

where μ is the gravitational constant, a_e is the equatorial radius of the Earth, and

$$P_{n,m} = (1-\varepsilon^2)^{m/2}\,\frac{\partial^m P_n}{\partial\varepsilon^m}$$

are the associated Legendre functions and P_n are the Legendre polynomials. Also, we have the sine of the latitude

$$\varepsilon = x_3/r$$

and the longitude is computed from

$$\lambda = \tan^{-1}(x_2/x_1)$$

For notational convention, we define a geopotential function U to be

$$U \equiv -V$$

and write

$$U = \frac{\mu}{r} + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\mu}{r}\left(\frac{a_e}{r}\right)^n P_{n,m}(\varepsilon)\,(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda) \tag{2}$$

$P_{n,m}$ becomes

$$P_{n,m} = \left(\frac{r^2 - x_3^2}{r^2}\right)^{m/2} \frac{\partial^m P_n}{\partial\varepsilon^m} = \frac{\rho^m}{r^m} P_n^m$$

where

$$\rho^2 \equiv x_1^2 + x_2^2$$

and

$$P_n^m \equiv \frac{\partial^m P_n}{\partial \varepsilon^m}$$

Also, define

$$C_m \equiv \rho^m \cos m\lambda$$

$$S_m \equiv \rho^m \cos m\lambda$$

$$B_{n,m} \equiv C_{n,m} C_m + S_{n,m} S_m$$

The geopotential can now be written

$$U = \frac{\mu}{r} + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\mu}{r} \left(\frac{a_e}{r}\right)^n \frac{P_n^m(\varepsilon)}{r^m} B_{n,m} \tag{3}$$

This form is especially useful since $P_n{}^m$, C_m, and S_m can be calculated recursively and the singularity at the pole ($\rho = 0$) can be avoided.

The Legendre polynomials may be calculated recursively as

$$P_n^m = P_{n-2}^m + (2n-1) P_{n-1}^{m-1} \ , \quad m \geq 1$$

$$P_n^0 = P_n = \Big((2n-1)\varepsilon P_{n-1} - (n-1) P_{n-2}\Big)/n$$

$$P_0^0 = 1 \ , \ P_0^1 = 0$$

$$P_1^0 = \varepsilon \ , \ P_1^1 = 1$$

also note that

$$C_m = \rho^m \cos m\lambda = \rho^{m-1+1} \cos(m-1+1)\lambda = C_1 C_{m-1} - S_1 S_{m-1}$$

$$S_m = \rho^m \sin m\lambda = \rho^{m-1+1} \sin(m-1+1)\lambda = S_1 C_{m-1} + C_1 S_{m-1}$$

$$C_1 = \rho \cos \lambda = \rho \, \frac{x_1}{\rho} = x_1$$

$$S_1 = \rho \sin \lambda = \rho \, \frac{x_2}{\rho} = x_2$$

It is helpful during coding to note that $\widetilde{C}_m \equiv C_m/r^m$ and $\widetilde{S}_m \equiv S_m/r^m$ are also recursive, since

$$\widetilde{C}_m = \frac{C_m}{r^m} = \frac{C_m}{r^{m-1+1}} = \widetilde{C}_1 \widetilde{C}_{m-1} - \widetilde{S}_1 \widetilde{S}_{m-1}$$

$$\widetilde{S}_m = \frac{S_m}{r^m} = \frac{S_m}{r^{m-1+1}} = \widetilde{S}_1 \widetilde{C}_{m-1} + \widetilde{C}_1 \widetilde{S}_{m-1}$$

$$\widetilde{C}_1 = \frac{x_1}{r} \ , \ \widetilde{S}_1 = \frac{x_2}{r}$$

SECTION 3
THE FIRST PARTIAL

The gravitational acceleration, $\underset{\sim}{g}$, is calculated as the first partial derivative of U with respect to the Earth-fixed position vector, $\underline{X}$. From equation (3), we have

$$\underset{\sim}{g} = \frac{\partial U}{\partial \underline{X}} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial \underline{X}} + \frac{\partial U}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \underline{X}} + \frac{\partial U}{\partial B_{n,m}}\frac{\partial B_{n,m}}{\partial \underline{X}} \tag{4}$$

where

$$\frac{\partial U}{\partial r} = -\frac{\mu}{r^2} - \sum\sum \frac{\mu}{r^2}\left(\frac{a_e}{r}\right)^n \frac{(n+m+1)}{r^m} P_n^m B_{n,m} \qquad \dagger$$

$$\frac{\partial U}{\partial \varepsilon} = \sum\sum \frac{\mu}{r}\left(\frac{a_e}{r}\right)^n P_n^{m+1} \frac{B_{n,m}}{r^m}$$

$$\frac{\partial U}{\partial B_{n,m}} = \sum\sum \frac{\mu}{r}\left(\frac{a_e}{r}\right)^n \frac{P_n^m}{r^m}$$

Using the recursive definition of C_m and S_m and the fact that

$$\tan\lambda = x_2/x_1$$

and

$$\underline{X}^T = (x_1\ x_2\ x_3)$$

it can now be shown that [1]

$$\frac{\partial B_{n,m}}{\partial \underline{X}} = \begin{bmatrix} m\,(C_{n,m}C_{m-1} + S_{n,m}S_{m-1}) \\ -m\,(C_{n,m}S_{m-1} - S_{n,m}C_{m-1}) \\ 0 \end{bmatrix} \equiv m \begin{bmatrix} B_{n,m-1} \\ -A_{n,m-1} \\ 0 \end{bmatrix} \equiv m\,\underline{b}$$

---

†For notational simplicity, $\sum\sum = \sum_{n-2}^{\infty}\sum_{m=0}^{n}$ throughout.

Also,

$$\frac{\partial r}{\partial \underline{X}} = \frac{\underline{X}}{r}$$

and,

$$\frac{\partial \varepsilon}{\partial \underline{X}} = \frac{1}{r} a - \frac{x_3}{r^3} \underline{X}$$

where $\qquad \underline{a}^T = (0 \quad 0 \quad 1)$

Combining these partials and substituting into equation (4), we get

$$\frac{\partial U}{\partial \underline{X}} = -\frac{\mu}{r^2}\left(1 + \sum\sum\left(\frac{a_e}{r}\right)^n (n+m+1)\,\frac{P_n^m B_{n,m}}{r^m}\right)\frac{\underline{X}}{r}$$

$$+ \frac{\mu}{r^2}\sum\sum\left(\frac{a_e}{r}\right)^m P_n^{m+1}\,\frac{B_{n,m}}{r^m}\,\underline{a} - \frac{\mu}{r^2}\sum\sum\left(\frac{a_e}{r}\right)^n P_n^{m+1}\,\frac{B_{n,m}}{r^m}\,\frac{x_3}{r}\,\frac{\underline{X}}{r}$$

$$+ \frac{\mu}{r^2}\sum\sum\left(\frac{a_e}{r}\right)^n \frac{P_n^m}{r^{m-1}}\begin{bmatrix} m\,(C_{n,m}C_{m-1} + S_{n,m}S_{m-1}) \\ -m\,(C_{n,m}S_{m-1} - S_{n,m}C_{m-1}) \\ 0 \end{bmatrix}$$

Defining,

$$J_n \equiv \sum_{m=1}^{n} m\,\frac{P_n^m}{r^{m-1}}\,(C_{n,m}C_{m-1} + S_{n,m}S_{m-1})$$

$$K_n \equiv -\sum_{m=1}^{n} m\,\frac{P_n^m}{r^{m-1}}\,(C_{n,m}S_{m-1} - S_{n,m}C_{m-1})$$

$$\Gamma_n \equiv C_{n,o}\,(n+1)\,P_n^o + \sum_{m=1}^{n}(1+n+m)\,\frac{P_n^m}{r^m}\,(C_{n,m}C_m + S_{n,m}S_m)$$

$$H_n \equiv C_{n,o}\,P_n^1 + \sum_{m=1}^{n}\frac{P_n^{m+1}}{r^m}\,(C_{n.m}C_m + S_{n,m}S_m)$$

$$H \equiv \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n H_n$$

$$\Gamma \equiv 1 + \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \Gamma_n$$

$$J \equiv \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n J_n$$

$$K \equiv \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n K_n$$

$$\Lambda \equiv \Gamma + \frac{x_3 H}{r}$$

$$\hat{X} \equiv \frac{\underline{X}}{r}$$

we can write

$$\underline{g} \equiv \frac{\partial U}{\partial \underline{X}} = -\frac{\mu}{r^2}\left(\Lambda \hat{X} - \begin{bmatrix} J \\ K \\ H \end{bmatrix}\right) \tag{5}$$

Note that the final result for the first partial derivative of the geopotential is a rather simple, compact, vector equation.

SECTION 4
THE SECOND PARTIAL

Next we calculate

$$\frac{\partial^2 U}{\partial \underline{X}^2}$$

starting with equation (4), as it leads directly to a compact symmetric notation.

We have from equation (4)

$$\frac{\partial U}{\partial \underline{X}} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial \underline{X}} + \frac{\partial U}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \underline{X}} + \frac{\partial U}{\partial B_{n,m}}\frac{\partial B_{n,m}}{\partial \underline{X}}$$

Thus,

$$\begin{aligned}\frac{\partial^2 U}{\partial \underline{X}^2} &= \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial \underline{X}}\frac{\partial r^T}{\partial \underline{X}} + \frac{\partial^2 U}{\partial \varepsilon^2}\frac{\partial \varepsilon}{\partial \underline{X}}\frac{\partial \varepsilon^T}{\partial \underline{X}} + \frac{\partial^2 U}{\partial B^2_{n,m}}\frac{\partial B_{n,m}}{\partial \underline{X}}\frac{\partial B^T_{n,m}}{\partial \underline{X}} \\ &+ \frac{\partial^2 U}{\partial \varepsilon \partial r}\left(\frac{\partial r}{\partial \underline{X}}\frac{\partial \varepsilon^T}{\partial \underline{X}} + \frac{\partial \varepsilon}{\partial \underline{X}}\frac{\partial r^T}{\partial \underline{X}}\right) + \frac{\partial^2 U}{\partial r \partial B_{n,m}}\left(\frac{\partial B_{n,m}}{\partial \underline{X}}\frac{\partial r^T}{\partial \underline{X}} + \frac{\partial r}{\partial \underline{X}}\frac{\partial B^T_{n,m}}{\partial \underline{X}}\right) \\ &+ \frac{\partial^2 U}{\partial \varepsilon \partial B_{n,m}}\left(\frac{\partial B_{n,m}}{\partial \underline{X}}\frac{\partial \varepsilon^T}{\partial \underline{X}} + \frac{\partial \varepsilon}{\partial \underline{X}}\frac{\partial B^T_{n,m}}{\partial \underline{X}}\right) + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial \underline{X}^2} + \frac{\partial U}{\partial \varepsilon}\frac{\partial^2 \varepsilon}{\partial \underline{X}^2} + \frac{\partial U}{\partial B_{n,m}}\frac{\partial^2 B_{n,m}}{\partial \underline{X}^2}\end{aligned} \tag{6}$$

The second and cross partials appearing in equation (6) are

$$\frac{\partial^2 U}{\partial r^2} = \frac{2\mu}{r^3} + \sum\sum \mu\left(\frac{a_e}{r}\right)^n \frac{(m+n+1)}{r^3}\frac{(m+n+2)}{r^m} P^m_n B_{n,m}$$

$$\frac{\partial^2 U}{\partial \varepsilon^2} = \sum\sum \frac{\mu}{r}\left(\frac{a_e}{r}\right)^n \frac{1}{r^m} P^{m+2}_n B_{n,m}$$

$$\frac{\partial^2 U}{\partial B_{n,m}} = 0$$

$$\frac{\partial^2 U}{\partial \varepsilon \partial r} = -\sum\sum \frac{\mu}{r^2}\left(\frac{a_e}{r}\right)^n \frac{(m+n+1)}{r^m} P^{m+1}_n B_{n,m}$$

$$\frac{\partial^2 U}{\partial \varepsilon \partial B_{n,m}} = \sum \sum \frac{\mu}{r} \left(\frac{a_e}{r} \right)^n \frac{P_n^{m+1}}{r^m}$$

$$\frac{\partial^2 U}{\partial r \partial B_{n,m}} = - \sum \sum \frac{\mu}{r^2} \left(\frac{a_e}{r} \right)^n \frac{(m + n + 1)}{r^m} P_n^m$$

$$\frac{\partial r}{\partial \underline{X}} \frac{\partial r^T}{\partial \underline{X}} = \frac{\underline{X}\,\underline{X}^T}{r^2}$$

$$\frac{\partial \varepsilon}{\partial \underline{X}} \frac{\partial \varepsilon^T}{\partial \underline{X}} = \frac{x_3^2}{r^6} \underline{X}\,\underline{X}^T - \frac{x_3}{r^4} (\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) + \frac{1}{r^2} \underline{a}\,\underline{a}^T$$

$$\frac{\partial^2 r}{\partial \underline{X}^2} = \frac{1}{r} \left(1 - \frac{\underline{X}\,\underline{X}^T}{r^2} \right)$$

$$\frac{\partial^2 \varepsilon}{\partial \underline{X}^2} = \frac{3x_3}{r^5} \underline{X}\,\underline{X}^T - \frac{1}{r^3} (\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) - \frac{x_3}{r^3} I$$

$$\frac{\partial^2 B_{n,m}}{\partial \underline{X}^2} = m(m-1) \begin{bmatrix} B_{n,m-2} & -A_{n,m-2} & 0 \\ -A_{n,m-2} & -B_{n,m-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$B_{n,m-2} \equiv C_{n,m} C_{m-2} + S_{n,m} S_{m-2}$$

$$A_{n,m-2} \equiv C_{n,m} S_{m-2} - S_{n,m} C_{m-2}$$

We also note the special combinations

$$\frac{\partial r}{\partial \underline{X}} \frac{\partial \varepsilon^T}{\partial \underline{X}} + \frac{\partial \varepsilon}{\partial \underline{X}} \frac{\partial r^T}{\partial \underline{X}} = \frac{1}{r^2} (\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) - 2 \frac{x_3}{r^4} \underline{X}\,\underline{X}^T$$

and

$$\frac{\partial B_{n,m}}{\partial \underline{X}} \frac{\partial r^T}{\partial \underline{X}} + \frac{\partial r}{\partial \underline{X}} \frac{\partial B^T_{n,m}}{\partial \underline{X}} = m \begin{bmatrix} B_{n,m-1} \\ -A_{n,m-1} \\ 0 \end{bmatrix} \frac{\underline{X}^T}{r} + m \frac{\underline{X}}{r} [B_{n,m-1} \quad -A_{n,m-1} \quad 0]$$

$$= \frac{m}{r} (\underline{b}\,\underline{X}^T + \underline{X}\,\underline{b}^T)$$

where

$$\underline{b} \equiv \begin{bmatrix} B_{n,m-1} \\ -A_{n,m-1} \\ 0 \end{bmatrix}$$

and

$$\frac{\partial B_{n,m}}{\partial \underline{X}} \frac{\partial \varepsilon^T}{\partial \underline{X}} + \frac{\partial \varepsilon}{\partial \underline{X}} \frac{\partial B^T_{n,m}}{\partial \underline{X}} = \frac{m}{r} \left[(\underline{b}\,\underline{a}^T + \underline{a}\,\underline{b}^T) - \frac{x_3}{r^2} (\underline{b}\,\underline{X}^T + \underline{X}\,\underline{b}^T) \right]$$

Putting all these into equation (6) leads to

$$\frac{\partial^2 U}{\partial \underline{X}^2} = \frac{\mu}{r^3} \left\{ 2 + \sum\sum \left(\frac{a_e}{r}\right)^n (m+n+1) \frac{(m+n+2)}{r^m} P^m_n B_{n,m} \right\} \frac{\underline{X}\,\underline{X}^T}{r^2}$$

$$+ \frac{\mu}{r^3} \sum\sum \left(\frac{a_e}{r}\right)^n P^{m+2}_n \frac{B_{n,m}}{r^m} \left[\frac{x_3^2}{r^4} \underline{X}\,\underline{X}^T \frac{x_3}{r^2} (\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) + \underline{a}\,\underline{a}^T \right]$$

$$- \frac{\mu}{r^3} \sum\sum \left(\frac{a_e}{r}\right)^n \frac{(n+m+1)}{r^m} P^{m+1}_n B_{n,m} \left[\frac{1}{r} (\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) - 2 \frac{x_3}{r^3} \underline{X}\,\underline{X}^T \right]$$

$$+ \frac{\mu}{r^3} \sum\sum \left(\frac{a_e}{r}\right)^n \frac{P^{m+1}_n}{r^{m-1}} m \left[(\underline{b}\,\underline{a}^T + \underline{a}\,\underline{b}^T) - \frac{x_3}{r^2} (\underline{b}\,\underline{X}^T + \underline{X}\,\underline{b}^T) \right]$$

$$- \frac{\mu}{r^3} \sum\sum \left(\frac{a_e}{r}\right)^n \frac{(n+m+1)}{r^{m-1}} P^m_n \frac{m}{r} \left[(b\,\underline{X}^T + \underline{X}\,b^T) \right]$$

$$- \frac{\mu}{r^3} \left(1 + \sum\sum \left(\frac{a_e}{r}\right)^n \frac{(n+m+1)}{r^m} P^m_n B_{n,m} \right) \left[I - \frac{\underline{X}\,\underline{X}^T}{r^2} \right]$$

$$+ \frac{\mu}{r^3} \sum\sum \left(\frac{a_e}{r}\right)^n P^{m+1}_n \frac{B_{n,m}}{r^m} \left[3 \frac{x_3}{r^3} \underline{X}\,\underline{X}^T - \frac{1}{r} (\underline{X}\,\underline{a}^T + \underline{a}\underline{X}^T) - \frac{x_3}{r} I \right]$$

$$+\frac{\mu}{r^3}\sum\sum\left(\frac{a_e}{r}\right)^n \frac{P_n^m}{r^{m-2}}\, m(m-1)\begin{bmatrix} B_{n,m-2} & -A_{n,m-2} & 0 \\ -A_{n,m-2} & -B_{n,m-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (7)$$

If we now define

$$L \equiv 2 + \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (n+m+1)(n+m+2)\, P_n^m \frac{B_{n,m}}{r^m}$$

$$M \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} \frac{P_n^{m+2}}{r^m} B_{n,m}$$

$$N \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=2}^{n} P_n^m\, m(m-1)\, \frac{(C_{n,m}C_{m-2} + S_{n,m}S_{m-2})}{r^{m-2}}$$

$$\Omega \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=2}^{n} P_n^m\, m(m-1)\, \frac{(C_{n,m}S_{m-2} - S_{n,m}C_{m-2})}{r^{m-2}}$$

$$P \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} P_n^{m+1}(m+n+1)\frac{B_{n,m}}{r^m}$$

$$Q \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} P_n^{m+1}\, m\, \frac{(C_{n,m}C_{m-1} + S_{n,m}S_{m-1})}{r^{m-1}}$$

$$R \equiv -\sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} P_n^{m+1}\, m\, \frac{(C_{n,m}S_{m-1} - S_{n,m}C_{m-1})}{r^{m-1}}$$

$$S \equiv \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} (m+n+1)\, P_n^m\, m\, \frac{(C_{n,m}S_{m-1} - S_{n,m}C_{m-1})}{r^{m-1}}$$

$$T \equiv -\sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \sum_{m=1}^{n} (m+n+1)\, P_n^m\, m\, \frac{(C_{n,m}S_{m-1} - S_{n,m}C_{m-1})}{r^{m-1}}$$

then with the definitions

$$\underline{\alpha} \equiv \begin{bmatrix} Q \\ R \\ 0 \end{bmatrix}$$

$$\underline{Y} \equiv \begin{bmatrix} S \\ T \\ 0 \end{bmatrix}$$

we can write equation (7) as

$$\begin{aligned}\frac{\partial^2 U}{\partial \underline{X}^2} &= \frac{\mu}{r^3} L \frac{\underline{X}\,\underline{X}^T}{r^2} + \frac{\mu}{r^3} M\left[\frac{x_3^2}{r^4}\underline{X}\,\underline{X}^T - \frac{x_3}{r^2}(\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) + \underline{a}\,\underline{a}^T\right] \\ &- \frac{\mu}{r^3} P\left[\frac{1}{r}(\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) - 2\frac{x_3}{r^3}\underline{X}\,\underline{X}^T\right] + \frac{\mu}{r^3}\left[(\underline{\alpha}\,\underline{a}^T + \underline{a}\,\underline{\alpha}^T) - \frac{x_3}{r^3}(\underline{\alpha}\,\underline{X}^T + \underline{X}\,\underline{\alpha}^T)\right] \\ &- \frac{\mu}{r^3}\left[\frac{\underline{Y}\,\underline{X}^T + \underline{X}\,\underline{Y}^T}{r}\right] - \frac{\mu}{r^3}\Gamma\left[I - \frac{\underline{X}\,\underline{X}^T}{r^2}\right] + \frac{\mu}{r^3} H\left[\frac{3x_3}{r^3}\underline{X}\,\underline{X}^T - \frac{1}{r}(\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T) - \frac{x_3}{r} I\right] \\ &+ \frac{\mu}{r^3}\begin{bmatrix} N & -\Omega & 0 \\ -\Omega & -N & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{8}$$

Collecting like terms, we get

$$\begin{aligned}\frac{\partial^2 U}{\partial \underline{X}^2} &= \frac{\mu}{r^3}\left(L + M\frac{x_3^2}{r^2} + 2P\frac{x_3}{r} + \Gamma + 3H\frac{x_3}{r}\right)\frac{\underline{X}\,\underline{X}^T}{r^2} \\ &- \frac{\mu}{r^3}\left(M\frac{x_3}{r} + P + H\right)\left[\frac{\underline{X}\,\underline{a}^T + \underline{a}\,\underline{X}^T}{r}\right] \\ &+ \frac{\mu}{r^3} M\underline{a}\,\underline{a}^T - \frac{\mu}{r^3}\left(\Gamma + \frac{x_3}{r}H\right) I \\ &+ \frac{\mu}{r^3}\left\{(\underline{\alpha}\,\underline{a}^T + \underline{a}\,\underline{\alpha}^T) - \frac{x_3}{r^2}(\underline{\alpha}\,\underline{X}^T + \underline{X}\,\underline{\alpha}^T) - \left(\frac{\underline{Y}\,\underline{X}^T + \underline{X}\,\underline{Y}^T}{r}\right) + \begin{bmatrix} N & -\Omega & 0 \\ -\Omega & -N & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}\end{aligned} \tag{9}$$

Recalling that

$$\Lambda = (\Gamma + \varepsilon H)$$

$$\hat{X} = \frac{\underline{X}}{r}$$

and defining

$$F \equiv L + \varepsilon(M\varepsilon + 2(P + H)) + \Lambda$$

$$G \equiv -(M\varepsilon + P + H)$$

$$\underline{d} \equiv \varepsilon\underline{a} + \underline{v}$$

we have finally

$$\frac{\partial^2 U}{\partial \underline{X}^2} = \frac{\mu}{r^3}\left\{[\hat{X} \vdots \underline{a}] \begin{bmatrix} F & G \\ G & M \end{bmatrix} \begin{bmatrix} \hat{X}^T \\ \hline \underline{a}^T \end{bmatrix} + [\hat{X} \vdots \underline{d}] \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{X}^T \\ \hline \underline{d}^T \end{bmatrix} + \begin{bmatrix} N - \Lambda & -\Omega & Q \\ -\Omega & -(N + \Lambda) & R \\ Q & R & -\Lambda \end{bmatrix} \right\} \tag{10}$$

Note that the final result for the second partial derivative of the geopotential is a rather simple, compact, symmetric matrix equation.

SECTION 5
NUMERICAL VERIFICATION

The equations in this report were coded and checked numerically using GOTPOT.

The computer program, GOTPOT, is given in the appendix in FORTRAN. Also included in the appendix is the truncated set of coefficients used in the numerical verification. The coefficients are contained in the subroutine, GEM10, which must be called before GOTPOT is called.

Two types of numerical checks were done. The first derives from the fact that U satisfies Laplace's equation; i.e., that

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = 0$$

Therefore, the trace of $\dfrac{\partial^2 U}{\partial \underline{X}^2}$ must be zero.

Secondly, numerical partials of $\underline{g}$ were taken by varying x_1, x_2, and x_3 ± 4 meters. The central difference derivatives were subtracted from the appropriate term in $\partial^2 U / \partial \underline{X}^2$ and then normalized by dividing the error by that term. The results of the numerical tests with $n = m = 4$ and $n = m = 5$ are given in table I.

Note that the trace is zero to 16 decimal digits compared to the individual diagonal elements. This is as close to zero as double precision arithmetic can get.

Note also that the numerical partials agree with the analytic to at least nine digits. These are inherently not as accurate as the trace since we are comparing a numerical partial to an analytic one. Nonetheless, the agreement is outstanding.

It should be noted that different computers may get slightly diferent results for the trace and error terms because we are dealing with numbers at the edge of computer accuracy.

These tests, together with comparisons of the formulae with direct differentiation of U considering only J_2 (not presented here) lead to the following.

SECTION 6
CONCLUSIONS

A non-singular, recursive algorithm for computing the first and a symmetric, non-singular, recursive algorithm for computing the second derivative of the geopotential has been derived, programmed, and verified.

SECTION 7
REFERENCE

1. Mueller, Alan C. "A Fast Recursive Algorithm for Calculating the Forces Due to the Geopotential (Program: GEOPOT)." JSC Internal Note 75-FM-42, June 1975.

TABLE I.- NUMERICAL VERIFICATION OF RESULTS

NOMINAL $\underline{X}$ = 5489150.0 , 802222.0 , 3140916.0 (meters)

ANALYTIC DGDX $n = 4, m = 4$

0.18777919E-05	0.49927074E-06	0.19651588E-05
0.49927074E-06	−0.14652000E-05	0.28721411E-06
0.19651588E-05	0.28721411E-06	−0.41259196E-06

TRACE 5.29395592033937712E-23

NUMERICAL DGDX $n = 4, m = 4$ ($dx = \pm 4$ meters)

0.18777919E-05	0.49927074E-06	0.19651588E-05
0.49927074E-06	−0.14652000E-05	0.28721411E-06
0.19651588E-05	0.28721411E-06	−0.41259196E-06

$(A - N)/A$

−0.39090978E-10	−0.30118256E-10	−0.44035904E-10
−0.25248719E-09	0.97483656E-11	0.45181657E-10
0.68954763E-10	−0.51455571E-10	−0.14525828E-09

ANALYTIC DGDX $n = 5, m = 5$

0.18777323E-05	0.49925937E-06	0.19650747E-05
0.49925937E-06	−0.14651356E-05	0.28720884E-06
0.19650747E-05	0.28720884E-06	−0.41259666E-06

TRACE 4.76456032830543941E-22

NUMERICAL DGDX $n = 5, m = 5$ ($dx = \pm 4$ meters)

0.18777323E-05	0.49925938E-06	0.19650747E-05
0.49925937E-06	−0.14651356E-05	0.28720884E-06
0.19650747E-05	0.28720884E-06	−0.41259666E-06

$(A - N)/A$

−0.28756763E-10	−0.18302270E-10	0.20757122E-10
0.48203922E-09	−0.75195398E-11	−0.16305638E-09
0.20757122E-10	−0.66417378E-10	0.23218316E-09

APPENDIX
COMPUTER LISTINGS

```
      SUBROUTINE GOTPOT(X,G,DGDX,NAX,MAX,ID)
C
C   PROGRAMMED BY ROBERT G. GOTTLIEB
C
C   NOTE: ID = 2 FOR SECOND PARTIALS, MUST = 1 OTHERWISE
C
      COMMON/GRCOEF/C(20),S(20),MU,RE
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION X(3),G(3),DGDX(3,3),PN(0:20),PNM1(0:20),PNM2(0:20)
      DIMENSION CTIL(0:20),STIL(0:20)
      DOUBLE PRECISION MU,MUOR2,MUOR3,MXPNM,LAMBDA
      R = DSQRT( X(1)**2 + X(2)**2 + X(3)**2 )
      RI = 1.D0/R
      XOR = X(1)*RI
      YOR = X(2)*RI
      ZOR = X(3)*RI
      EP = ZOR
      REOR = RE*RI
      REORN = REOR
      MUOR2 = MU*RI*RI
      K = 0
      DO 10 M = 1,MAX + ID
      PNM2(M) = 0.D0
   10 PNM1(M) = 0.D0
      PNM2(0) = 1.D0
      PNM1(0) = EP
      PNM1(1) = 1.D0
      CTIL(0) = 1.D0
      STIL(0) = 0.D0
      CTIL(1) = XOR
      STIL(1) = YOR
      SUMH = 0.D0
      SUMGM = 1.D0
      SUMJ = 0.D0
      SUMK = 0.D0
      IF(ID.EQ.2)THEN
       SUML = 2.D0
       SUMM = 0.D0
       SUMN = 0.D0
       SUMO = 0.D0
       SUMP = 0.D0
       SUMQ = 0.D0
       SUMR = 0.D0
       SUMS = 0.D0
       SUMT = 0.D0
      ENDIF
C
      DO 50 N = 2,NAX
```

```
      REORN = REORN*REOR
      N2M1 = N + N-1
      NM1 = N-1
      NP1 = N + 1
C ****************   RECURSIVE COMPUTATION OF LEGENDRE POLYNOMIALS - PN(0)
C ****************   AND FIRST ASSOCIATED LEGENDRE FUNCTIONS PN(1), PN (2)
      PN(0) = (N2M1*EP*PNM1(0)-NM1*PNM2(0))/N
      PN(1) = PNM2(1) + N2M1*PNM1(0)
      PN(2) = PNM2(2) + N2M1*PNM1(1)
      K = K + 1
      SUMHN = PN(1)*C(K)
      SUMGMN = PN(0)*C(K)*NP1
      IF(ID.EQ.2)THEN
       SUMLN = SUMGMN*(NP1 + 1)
       SUMMN = PN(2)*C(K)
       SUMPN = SUMHN*NP1
      ENDIF
      IF(MAX.GT.0) THEN
       SUMJN = 0.D0
       SUMKN = 0.D0
       IF(ID.EQ.2)THEN
        SUMNN = 0.D0
        SUMON = 0.D0
        SUMQN = 0.D0
        SUMRN = 0.D0
        SUMSN = 0.D0
        SUMTN = 0.D0
       ENDIF
       CTIL(N) = CTIL(1)*CTIL(NM1)-STIL(1)*STIL(NM1)
       STIL(N) = STIL(1)*CTIL(NM1) + CTIL(1)*STIL(NM1)
       IF(N.LT.MAX)THEN
         LIM = N
       ELSE
         LIM = MAX
       ENDIF
       DO 20 M = 1,LIM
       MM1 = M-1
       MP1 = M + 1
C *******   RECURSIVE COMPUTATION OF ASSOCIATED LEGENDRE FUNCTIONS - PN(M + 1)
       PN(MP1) = PNM2(MP1) + N2M1*PNM1(M)
       MXPNM = M*PN(M)
       NM = K + M
       BNMTIL = C(NM)*CTIL(M  ) + S(NM)*STIL(M  )
       BNMTM1 = C(NM)*CTIL(MM1) + S(NM)*STIL(MM1)
       ANMTM1 = C(NM)*STIL(MM1)-S(NM)*CTIL(MM1)
       SUMHN = SUMHN + PN(MP1)*BNMTIL
       SUMGMN = SUMGMN + (N + MP1)*PN(M)*BNMTIL
       SUMJN = SUMJN + MXPNM*BNMTM1
       SUMKN = SUMKN-MXPNM*ANMTM1
       IF(ID.EQ.2)THEN
         MP2 = M + 2
         NPMP1 = N + MP1
```

```
         PN(MP2) = PNM2(MP2) + N2M1*PNM1(MP1)
         SUMLN = SUMLN + NPMP1*(MP1 + NP1)*PN(M)*BNMTIL
         SUMMN = SUMMN + PN(MP2)*BNMTIL
         SUMPN = SUMPN + NPMP1*PN(MP1)*BNMTIL
         SUMQN = SUMQN + M*PN(MP1)*BNMTM1
         SUMRN = SUMRN-M*PN(MP1)*ANMTM1
         SUMSN = SUMSN + NPMP1*MXPNM*BNMTM1
         SUMTN = SUMTN-NPMP1*MXPNM*ANMTM1
         IF(M.GE.2)THEN
          MM2 = M-2
          SUMNN = SUMNN + MM1*MXPNM*(C(NM)*CTIL(MM2) + S(NM)*STIL(MM2))
          SUMON = SUMON + MM1*MXPNM*(C(NM)*STIL(MM2)-S(NM)*CTIL(MM2))
         END IF
       END IF
   20  CONTINUE
       SUMJ = SUMJ + REORN*SUMJN
       SUMK = SUMK + REORN*SUMKN
        IF(ID.EQ.2)THEN
         SUMN = SUMN + REORN*SUMNN
         SUMO = SUMO + REORN*SUMON
         SUMQ = SUMQ + REORN*SUMQN
         SUMR = SUMR + REORN*SUMRN
         SUMS = SUMS + REORN*SUMSN
         SUMT = SUMT + REORN*SUMTN
        ENDIF
      ENDIF
C ***************************  SUMS BELOW HERE HAVE VALUES WHEN M = 0
      SUMH = SUMH + REORN*SUMHN
      SUMGM = SUMGM + REORN*SUMGMN
      IF(ID.EQ.2)THEN
       SUML = SUML + REORN*SUMLN
       SUMM = SUMM + REORN*SUMMN
       SUMP = SUMP + REORN*SUMPN
      ENDIF
      K = K + N
C ************************** SHIFT LEGENDRE POLYNOMIALS DOWN
      IF(N.LT.NAX)THEN
       DO 40 J = 0,N
       PNM2(J) = PNM1(J)
       PNM1(J) = PN(J)
  40  CONTINUE
      ENDIF
  50 CONTINUE
C
      LAMBDA = SUMGM + EP*SUMH
      G(1) = -MUOR2*(LAMBDA*XOR-SUMJ)
      G(2) = -MUOR2*(LAMBDA*YOR-SUMK)
      G(3) = -MUOR2*(LAMBDA*ZOR-SUMH)
      IF(ID.EQ.2)THEN
C *************************** NEED TO CONSTRUCT 2ND PARTIAL DERIVATIVE MATRIX
       GG = -(SUMM*EP + SUMP + SUMH)
       FF = SUML + LAMBDA + EP*(SUMP + SUMH - GG)
```

```
 D1 = EP*SUMQ + SUMS
 D2 = EP*SUMR + SUMT
 MUOR3 = MUOR2*RI
 DGDX(1,1) = MUOR3*((FF*XOR-2.D0*D1)*XOR-LAMBDA + SUMN)
 DGDX(2,2) = MUOR3*((FF*YOR-2.D0*D2)*YOR-LAMBDA-SUMN)
 DGDX(3,3) = MUOR3*((FF*ZOR + 2.D0*GG)*ZOR-LAMBDA + SUMM)
 DGDX(1,2) = MUOR3*((FF*YOR-D2)*XOR-D1*YOR-SUMO)
 DGDX(1,3) = MUOR3*((FF*XOR-D1)*ZOR + GG*XOR + SUMQ)
 DGDX(2,3) = MUOR3*((FF*YOR-D2)*ZOR + GG*YOR + SUMR)
 DGDX(2,1) = DGDX(1,2)
 DGDX(3,1) = DGDX(1,3)
 DGDX(3,2) = DGDX(2,3)
ENDIF
RETURN
END
```

```
      SUBROUTINE GEM10

      COMMON/GRCOEF/C(20),S(20),MU,RE
      DOUBLE PRECISION MU,CC(5,0:5),SS(5,0:5)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

*     The constants given herein are taken from
*
*     "Gravity Model Improvement Using Geos 3 (GEM 9 and GEM 10)"
*                 by
*     F. Lerch, S. Klosko, R. Laubscher, and C. Wagner
*     Journal of Geophysical Research, Vol. 84, N0. B8, July 30, 1979

*     From Table 8 --Comparison of the IAG 1975 and GEM Geodetic Parameters
      MU = 39860047.0D7
      RE = 6378139.0D0

*     From Table 4--Goddard Earth Model 10 Normalized Coefficients
*     (We use an abbreviated set for reasons of space)
      CC( 2, 0) = -484.16544D0
      CC( 3, 0) =    0.95838D0
      CC( 4, 0) =    0.54112D0
      CC( 5, 0) =    0.06862D0

      CC( 2, 1) =    0.00104D0
      SS( 2, 1) =   -0.00243D0
      CC( 3, 1) =    2.02855D0
      SS( 3, 1) =    0.25197D0
      CC( 4, 1) =   -0.53521D0
      SS( 4, 1) =   -0.46926D0
      CC( 5, 1) =   -0.05117D0
      SS( 5, 1) =   -0.09379D0

      CC( 2, 2) =    2.43404D0
      SS( 2, 2) =   -1.39907D0
      CC( 3, 2) =    0.89272D0
      SS( 3, 2) =   -0.62346D0
      CC( 4, 2) =    0.35208D0
      SS( 4, 2) =    0.66404D0
      CC( 5, 2) =    0.65146D0
      SS( 5, 2) =   -0.32769D0

      CC( 3, 3) =    0.70028D0
      SS( 3, 3) =    1.41250D0
      CC( 4, 3) =    0.98850D0
      SS( 4, 3) =   -0.20179D0
      CC( 5, 3) =   -0.46712D0
      SS( 5, 3) =   -0.20298D0

      CC( 4, 4) =   -0.19531D0
      SS( 4, 4) =    0.29883D0
      CC( 5, 4) =   -0.28754D0
```

```
      SS( 5, 4) =   0.04990D0

      CC( 5, 5) =   0.15617D0
      SS( 5, 5) =  -0.65983D0

* The unnormalized coefficients, in single subscript form, used in GOTPOT
* are computed below.

      DO 200 N = 2,5
      DO 100 M = 0,N
      J = (N**2 + N -4)/2 + M
      IF( M .EQ. 0 ) THEN
       A = 2*N + 1
       C(J) = DSQRT(A)*CC(N,0)*1.0D-6
       S(J) = 0.D0
      ELSE
       COEF = 2*(2*N + 1)*FACTNK(N-M,N + M)
       COEF = DSQRT(COEF)*1.0D-6
       C(J) = COEF * CC(N,M)
       S(J) = COEF * SS(N,M)
      END IF
  100 CONTINUE
  200 CONTINUE
      RETURN
      END


      DOUBLE PRECISION FUNCTION FACTNK(N,K)
      DOUBLE PRECISION ANSWER
*     This routine computes n!/k!
      INTEGER START,STOP
      IF(N.GT.K)THEN
       START = K + 1
       STOP = N
      ELSE IF(N.EQ.K)THEN
       FACTNK = 1.D0
       RETURN
      ELSE
       START = N + 1
       STOP = K
      END IF
      ANSWER = START
      DO 100 I = START + 1,STOP
       ANSWER = ANSWER*I
  100 CONTINUE
      IF(K.GT.N)ANSWER = 1.D0/ANSWER
      FACTNK = ANSWER
      RETURN
      END
```

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